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Non-integrable representations of the restricted quantum analogue of $sl(3)$

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Abstract. The structure of irreducible representations of (restricted) $\mathcal{U}_q(sl(3))$ at roots of unity is understood within the Gelfand–Zetlin basis. The latter needs a weakened definition for non-integrable representations, where the quadratic Casimir operator of the quantum subalgebra $\mathcal{U}_q(sl(2)) \subset \mathcal{U}_q(sl(3))$ is not completely diagonalized. This is necessary in order to take into account the indecomposable $\mathcal{U}_q(sl(2))$ -modules that appear. The set of redefined (mixed) states has a teepee shape inside the pyramid made with the whole representation.

1. Introduction

In this paper, we are interested in finite-dimensional representations of the quantum analogue of the enveloping algebra of $sl(3)$ at roots of unity, in the restricted specialization.

When the deformation parameter, q , is not a root of unity, the finite-dimensional irreducible representations of quantum groups, as defined in [1, 2], are in correspondence with the classical ones [3, 4]. This correspondence is 2^{rank} -to-one, the factor 2^{rank} being related to trivial isomorphisms of the quantum enveloping algebra.

When q is a root of unity, the dimension of finite-dimensional irreducible representations is bounded. In the unrestricted specialization, new classes of irreducible representations appear that are characterized by continuous parameters (see [5] for $\mathcal{U}_q(sl(2))$ and [6] for general $\mathcal{U}_q(\mathcal{G})$). We do not consider them here, since we are interested in the restricted specialization, and more precisely in its finite-dimensional Hopf subalgebra, where the raising and lowering generators are nilpotent and where the Cartan generators are quantized. In this case, the finite-dimensional irreducible representations can be obtained as a quotient of Verma modules (with integral dominant highest weights) by their maximal submodule.

As for representations of Lie algebras in finite characteristics, the irreducible representation corresponding to a given highest weight that may have a smaller dimension than the classical one [7–10].

Another feature can arise for $\mathcal{U}_q(sl(N))$ representations in the limit when $q^l = 1$: they can be *non-integrable*, in the sense that the $\mathcal{U}_q(sl(N-1)) \subset \mathcal{U}_q(sl(N))$ representations it contains may become indecomposable.

In the classical case and in the case of generic q , the Gelfand–Zetlin (G–Z) basis for $\mathcal{U}_q(sl(3))$ irreducible representations simultaneously diagonalizes the Cartan generators and the quadratic Casimir operator $C_{\mathcal{U}_q(sl(2))}$ of $\mathcal{U}_q(sl(2))$ [11].

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Some pictures are available at <http://lapphp0.in2p3.fr/~arnaudon/teepee.html>

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When q is a root of unity, some of the $\mathcal{U}_q(\mathfrak{sl}(2))$ -modules involved in a simple $\mathcal{U}_q(\mathfrak{sl}(3))$ -module can be indecomposable with a non-diagonalizable action of the quadratic Casimir operator $C_{\mathcal{U}_q(\mathfrak{sl}(2))}$. If the definition of the G–Z basis includes the requirement that $C_{\mathcal{U}_q(\mathfrak{sl}(2))}$ is diagonalized, then this basis cannot exist for such a representation. If we consider the weaker requirement that $C_{\mathcal{U}_q(\mathfrak{sl}(2))}$ is expressed in indecomposable blocks, then the G–Z basis exists, as we will show. Indeed, in the limit when q is a root of unity of the order of l , the $\mathcal{U}_q(\mathfrak{sl}(2))$ representations of dimensions $l + d$ and $l - d$ have the same value of $C_{\mathcal{U}_q(\mathfrak{sl}(2))}$ and they are coupled in a single indecomposable representation.

The signal that the G–Z basis without modification does not work for non-integrable irreducible restricted representations at roots of unity is given by the fact that some denominators vanish in the coefficients. No scale change can solve this problem. As explained in [12], a solution to cure the divergences is to suitably mix the states with the same quantum numbers. Since this mixing involves zero or infinite coefficients, the correct way is to perform it at generic q and to take the limit. The limit of *all* matrix elements being zero or finite, we get a well-defined description of the representation at $q^l = 1$.

With such a well-defined description, it is then possible to exhibit the subrepresentation, in the cases when it exists.

The results of this paper may be summarized as follows.

- In the classical case, or when q is generic, the total $\mathcal{U}_q(\mathfrak{sl}(2))$ representation corresponding to a given value of the Cartan element $h_1 + 2h_2$ that commutes with $\mathcal{U}_q(\mathfrak{sl}(2))$ is equivalent to the tensor product of two $\mathcal{U}_q(\mathfrak{sl}(2))$ irreducible representations, the rule is shown in figure 3. When $q^l = 1$, this property remains true, but the tensor product now decomposes into indecomposable and irreducible representations.

- If we introduce the two transformations acting on G–Z states (the definitions are given in the next section)

$$S_1 : \left| \begin{array}{ccccc} p_{13} & & p_{23} & & p_{33} \\ & p_{12} & & p_{22} & \\ & & p_{11} & & \end{array} \right\rangle \mapsto \left| \begin{array}{ccccc} p_{13} & & p_{23} & & p_{33} \\ & p_{22} + l & & p_{12} - l & \\ & & p_{11} & & \end{array} \right\rangle \quad (1)$$

$$S_2 : V(p_{33} + l, p_{23}, p_{13} - l) \longrightarrow V(p_{13}, p_{23}, p_{33})$$

$$\left| \begin{array}{ccccc} p_{33} + l & & p_{23} & & p_{13} - l \\ & p_{12} & & p_{22} & \\ & & p_{11} & & \end{array} \right\rangle \mapsto \left| \begin{array}{ccccc} p_{13} & & p_{23} & & p_{33} \\ & p_{12} & & p_{22} & \\ & & p_{11} & & \end{array} \right\rangle. \quad (2)$$

S_2 is defined when $p_{33} + l > p_{23} > p_{13} - l$ only, then

—if a state *and* its image by S_1 belong to the representation, then these two states belong to the same (indecomposable $\mathcal{U}_q(\mathfrak{sl}(2))$) representation and should be redefined. The set of redefined states looks like a teepee or a tent, depending on the highest weight (see figure 1). This happens when the highest weight λ is such that $\langle \lambda, \theta^\vee \rangle \geq l$, where θ is the longest root.

—the image of S_2 is a subrepresentation. This image is exactly the subrepresentation described in [10]. This happens when the highest weight λ is such that $\langle \lambda + \rho, \theta^\vee \rangle > l$ and when its image by the reflection with respect to the line $\langle \lambda + \rho, \theta^\vee \rangle = l$ is also a dominant weight (ρ being the sum of fundamental weights).

The transformations S_1 and S_2 are particular cases of transformations introduced in [13] for periodic representations, corresponding to (i) symmetry among the G–Z indices of the same line: permutations of these indices leave the coefficients invariant (ii) invariance under a translation by l of a G–Z index. These symmetries become a problem for the restricted representations we consider here. The mixing and normalization of states that we introduce actually break them. The transformation S_2 now defines an isomorphism from

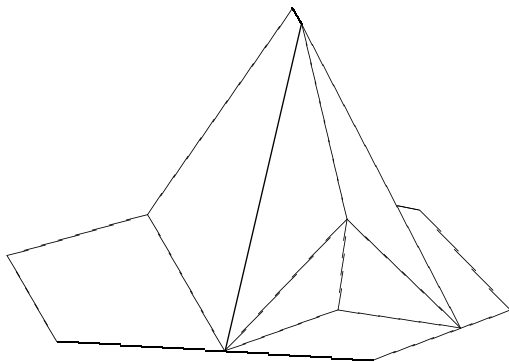


Figure 1. Teepee

$M_q(p_{33} + l, p_{23}, p_{13} - l)$ to a subrepresentation of $M_q(p_{13}, p_{23}, p_{33})$.

The structure of this paper is as follows. In section 2, we recall the definition of the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(3))$ and give the expression of the G–Z basis for generic deformation parameter q . In section 3, we propose a mixing of some states of the G–Z basis that allows a well-defined limit when $q^l = 1$. In section 4, the subrepresentation of the regularized representation is exhibited, when it exists. Finally, some technical expressions are given in appendices A and B for indecomposable $\mathcal{U}_q(\mathfrak{sl}(2))$ representations and for action within the set of redefined states. Final checks of finiteness of coefficients are made in appendix C.

2. Definitions

Let $\mathcal{U}_q(\mathfrak{sl}(3))$ be the unital algebra generated by e_i, f_i and h_i ($i = 1, 2$) with the relations

$$\begin{aligned}
 [h_i, e_j] &= a_{ij}e_j & [h_i, f_j] &= -a_{ij}f_j \\
 [e_i, f_j] &= \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} = \delta_{ij}[h_i] \\
 e_i^2 e_{i\pm 1} - (q + q^{-1})e_i e_{i\pm 1} e_i + e_{i\pm 1} e_i^2 &= 0 \\
 f_i^2 f_{i\pm 1} - (q + q^{-1})f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 &= 0
 \end{aligned} \tag{3}$$

where $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is the Cartan matrix of $\mathfrak{sl}(3)$. We define q -numbers by $[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}$.

Let α_1, α_2 be the simple roots, ω_1, ω_2 the fundamental weights, $P = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ the weight lattice and $\alpha_1^\vee, \alpha_2^\vee$ the co-roots, with $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

The longest root is $\theta = \alpha_1 + \alpha_2$. The sum of the fundamental weights (equal to half the sum of positive roots) is $\rho = \alpha_1 + \alpha_2$.

We will later be interested in the root of unity case. Let $l > 2$ be an odd integer. When $q^l = 1$, $(q^{h_i})^l$ is central and we will add the relations corresponding to the restricted

specialization

$$\begin{aligned} e_1^l = e_2^l = e_3^l = 0 & \quad \text{with } e_3 = e_1 e_2 - q^{-1} e_2 e_1 \\ f_1^l = f_2^l = f_3^l = 0 & \quad \text{with } f_3 = f_2 f_1 - q f_1 f_2 \\ q_i^{2h_i} = 1. \end{aligned} \quad (4)$$

These relations define a co-ideal with respect to the Hopf structure, so that quotienting by them leads to a Hopf algebra. We do not introduce the divided powers of the generators. The generators e_i , f_i and k_i and relations (3), (4) actually define a finite-dimensional Hopf subalgebra of the usual restricted specialization. As proved in [7], the study of finite-dimensional representations of the restricted specialization can be reduced to the study of those of finite subalgebra.

The finite-dimensional irreducible representations are labelled by integral dominant weights $\lambda = \lambda_1 \omega_1 + \lambda_2 \omega_2$, with $\lambda_i \in \mathbb{Z}_+$. We can limit the study to $0 \leq \lambda_i < l$ since translations of the highest weight by multiples of $l\omega_i$ provide equivalent representations (strictly speaking, the representations are only equivalent as representations of the algebra generated by e_i , f_i and q^{h_i} ($i = 1, 2$); a global translation of the weights is, however, the only difference).

As in the case of affine Lie algebras, a representation M is called integrable (see, e.g. [14]) if

- (i) $M = \bigoplus_{\Lambda \in P} M_\Lambda$, i.e. M is the direct sum of its weight spaces (common eigenspaces of the Cartan generators), the weights being integral (belonging to the weight lattice P),
- (ii) $\dim M_\Lambda < \infty$, i.e. each weight space has a finite dimension,
- (iii) M decomposes into a direct sum of finite-dimensional representations of the $\mathcal{U}_q(\mathfrak{sl}(2))$ subalgebras generated by e_i , f_i , q^{h_i} , q^{-h_i} for $i = 1, 2$.

In the case we consider, all irreducible representations have a finite dimension, since, at roots of unity, the quantum algebra is a finite-dimensional module over its centre. The first two requirements for integrability are hence always satisfied for irreducible representations. As we will see, the third one is not always satisfied since M may contain indecomposable representations of its quantum subalgebras.

For generic q , any finite-dimensional irreducible representation can be described using the G–Z basis. Let $M_q(p_{13}, p_{23}, p_{33})$ be the representation with the highest weight $(\lambda_1, \lambda_2) = (p_{13} - p_{23} - 1, p_{23} - p_{33} - 1)$, (the eigenvalues of h_1 and h_2 on the highest weight vector). It acts on the vector space $V(p_{13}, p_{23}, p_{33})$ of dimension

$$d(p_{13}, p_{23}, p_{33}) = \frac{1}{2}(p_{13} - p_{23})(p_{23} - p_{33})(p_{13} - p_{33}) \quad (5)$$

and spanned by vectors

$$\left| \begin{array}{ccc} p_{13} & p_{23} & p_{33} \\ & p_{12} & p_{22} \\ & & p_{11} \end{array} \right\rangle \quad (6)$$

with $p_{ij} \in \mathbb{Z}$, such that

$$p_{13} \geq p_{12} > p_{23} \geq p_{22} > p_{33} \quad p_{12} \geq p_{11} > p_{22}. \quad (7)$$

All the p_{ij} are defined up to an overall constant. The only differences are involved in the matrix elements. We use $p_{ij} = h_{ij} - i$ instead of the standard h_{ij} to make the symmetries among the indices of the same line more explicit. The first line of indices is constant for a given representation. We will sometimes omit it, such as when no confusion is possible and when it is the same as in (6).

The representations $M_q(p_{13}, p_{23}, p_{33})$ described here are actually in one-to-one correspondence with the classical ones. To get all the $2^{\text{rank}} = 4$ inequivalent representations corresponding to a classical, one can add to the index p_{11} , or to the index p_{12} , or to both of them, the constant $i\pi/\ln q$ (or $l/2$ if $q^l = 1$).

At generic q as well as in the classical case, the G–Z basis expresses the $\mathcal{U}_q(sl(3))$ representation as a direct sum of $\mathcal{U}_q(sl(2))$ irreducible representations (corresponding to fixed values of p_{12} and p_{22}). By $\mathcal{U}_q(sl(2))$, we will always mean the subalgebra of $\mathcal{U}_q(sl(3))$ generated by e_1, f_1, k_1 .

As in [10], we shall depict the set of G–Z state in a three-dimensional pyramid, with one point for each vector of the basis. The horizontal coordinates, x, y , are simply the values of the orthogonal Cartan elements h_1 and $h_1 + 2h_2$. The third coordinate, z , starts from zero and increases when the dimension $p_{12} - p_{22}$ of the $\mathcal{U}_q(sl(2))$ representation decreases.

$$\begin{aligned} x &= 2p_{11} - (p_{12} + p_{22}) - 1 \\ y &= 3(p_{12} + p_{22}) - 2(p_{13} + p_{23} + p_{33}) - 1 \\ z &= \min(p_{13} - p_{12}, p_{23} - p_{33} - 1). \end{aligned} \tag{8}$$

The actions of the generators on the G–Z basis are given by

$$h_1|p\rangle = (2p_{11} - (p_{12} + p_{22}) - 1)|p\rangle \tag{9}$$

$$h_2|p\rangle = (2(p_{12} + p_{22}) - p_{11} - (p_{13} + p_{23} + p_{33}) - 1)|p\rangle$$

$$f_1|p\rangle = ([p_{12} - p_{11} + 1][p_{11} - p_{22} - 1])^{1/2} \begin{vmatrix} p_{12} & & p_{22} \\ & p_{11} - 1 & \\ & & \end{vmatrix} \tag{10}$$

$$e_1|p\rangle = ([p_{12} - p_{11}][p_{11} - p_{22}])^{1/2} \begin{vmatrix} p_{12} & & p_{22} \\ & p_{11} + 1 & \\ & & \end{vmatrix}$$

$$f_2|p\rangle = \left(\frac{P_1 P_2}{P_3}(1, 2; p)\right)^{1/2} |p_{12} - 1\rangle + \left(\frac{P_1 P_2}{P_3}(2, 2; p)\right)^{1/2} |p_{22} - 1\rangle \tag{11}$$

$$e_2|p\rangle = \left(\frac{P_1 P_2}{P_3}(1, 2; p_{12} + 1)\right)^{1/2} |p_{12} + 1\rangle + \left(\frac{P_1 P_2}{P_3}(2, 2; p_{22} + 1)\right)^{1/2} |p_{22} + 1\rangle$$

where

$$\begin{aligned} P_1(1, 2; p) &= [p_{13} - p_{12} + 1][p_{12} - p_{23} - 1][p_{12} - p_{33} - 1] \\ P_1(2, 2; p) &= [p_{13} - p_{22} + 1][p_{23} - p_{22} + 1][p_{22} - p_{33} - 1] \\ P_2(1, 2; p) &= [p_{12} - p_{11}] \\ P_2(2, 2; p) &= [p_{11} - p_{22}] \\ P_3(1, 2; p) &= [p_{12} - p_{22}][p_{12} - p_{22} - 1] \\ P_3(2, 2; p) &= [p_{12} - p_{22}][p_{12} - p_{22} + 1] \end{aligned} \tag{12}$$

where p stands for the set of indices p_{ij} , and where $p_{ij} \pm 1$ in an argument shows the modified index only. The two first arguments i, j of the coefficients P_α indicate which p_{ij} is changed.

For generic q , the q -integers involved in the coefficients vanish only at zero argument. Vanishing denominators are compensated by two vanishing numerators.

When q goes to a primitive l th root of one, with l odd, the q -integer $[n]$ goes to zero iff n is a multiple of l . For this reason, new zeros arise in the denominator when $p_{13} - p_{33} - 2 \geq l$, i.e. when the highest weight satisfies $\langle \lambda, \theta^\vee \rangle = \lambda_1 + \lambda_2 \geq l$. These new

zeros are generally not compensated in the numerator. The previously defined G–Z basis is then not well-defined in this case.

When $\langle \lambda, \theta^\vee \rangle = p_{13} - p_{33} - 2 < l$, the representation is correctly described by the G–Z basis. The $\mathcal{U}_q(\mathfrak{sl}(2))$ representations it involves are completely reducible into irreducible representations of dimensions less than l .

The remaining case is then $p_{13} - p_{33} - 2 \geq l$ and still $p_{13} - p_{23} \leq l$ and $p_{23} - p_{33} \leq l$. It is the aim of section 3 to get a well-defined description of this case.

3. Regularization

We still consider a representation $M_q(p_{13}, p_{23}, p_{33})$ of $\mathcal{U}_q(\mathfrak{sl}(3))$ at generic q . Let $l > 2$ be an odd integer. When $p_{13} - p_{33} > l$ (and $p_{13} - p_{23} \leq l$, $p_{23} - p_{33} \leq l$), in prevision of the case $q^l = 1$, we perform the following transformation that depends on l .

Let us consider $\left| \begin{array}{cc} p_{12} & p_{22} \\ p_{11} & \end{array} \right\rangle$ with $p_{12} - p_{22} > l$. When both (7) and

$$\begin{aligned} p_{13} &\geq p_{22} + l > p_{23} \geq p_{12} - l > p_{33} \\ p_{22} + l &\geq p_{11} > p_{12} - l \end{aligned} \quad (13)$$

are satisfied, i.e. if the image of $\left| \begin{array}{cc} p_{12} & p_{22} \\ p_{11} & \end{array} \right\rangle$ by S_1 defined by (2) also belongs to the representation, we define

$$\left(\left| \begin{array}{cc} p_{12} & p_{22} \\ p_{11}' & \end{array} \right\rangle \right) = \left(\begin{array}{cc} [l]^{1/2} & 0 \\ \left(\frac{[l-1]}{[l+1][l]}\right)^{1/2} & \frac{1}{[l]^{1/2}} \end{array} \right) \left(\left| \begin{array}{cc} p_{12} & p_{22} \\ p_{22} + l & p_{12} - l \\ p_{11}' & \end{array} \right\rangle \right). \quad (14)$$

This transformation is inspired by that introduced in [12]. As in this paper, the transformation matrix has determinant 1 and its eigenvalues go to 0 and ∞ in the limit when $q^l = 1$. In the following, we keep the primed states and forget the corresponding unprimed states.

The set of G–Z states such that their image by S_1 is still a G–Z state (i.e. satisfying both (7) and (13)) is displayed (on the hexagonal basis of the pyramid) in figure 1. It looks like a teepee or a tent, depending on the values of p_{i3} . It includes both the redefined states and the $\mathcal{U}_q(\mathfrak{sl}(2))$ representations invariant under S_1 , with dimension $p_{12} - p_{22} = l$, that are not redefined.

3.1. Indecomposable $\mathcal{U}_q(\mathfrak{sl}(2))$ subrepresentations

The two $\mathcal{U}_q(\mathfrak{sl}(2))$ representations corresponding to $p_{12} \geq p_{11} > p_{22}$ (of dimension $p_{12} - p_{22}$) and $p_{22} + l \geq p_{11} > p_{12} - l$ (of dimension $2l - (p_{12} - p_{22})$) are gathered into a sum of dimension $2l$. In the limit when $q^l = 1$ this sum becomes indecomposable; it is described in figure 2. The actions of the generators e_1 and f_1 on this indecomposable representation are given in appendix A. Initially, e_1 and f_1 induced only moves along the x -direction. Now, the primed states replace the unprimed states inside the teepee, and e_1 and f_1 induce moves along the x -direction and possibly shortcut down in the z -direction. In the extreme case, this shortcut can lead directly from top to bottom.

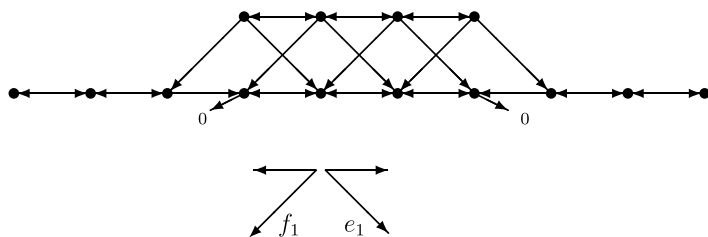


Figure 2. Indecomposable $\mathcal{U}_q(\mathfrak{sl}(2))$ representation with $q^7 = 1$ and $p_{12} - p_{22} = 10$.

The quadratic Casimir operator of $\mathcal{U}_q(\mathfrak{sl}(2))$ acts on the space spanned by $\begin{pmatrix} p_{12} & p_{22} \\ p_{11} & \end{pmatrix}'$ and $\begin{pmatrix} p_{22} + l & p_{12} - l \\ p_{11} & \end{pmatrix}'$ as the non-diagonalizable matrix

$$C_{\mathcal{U}_q(\mathfrak{sl}(2))} = \begin{pmatrix} q^{p_{12}-p_{22}} + q^{p_{22}-p_{12}} & 0 \\ i(q - q^{-1})^2[p_{12} - p_{22}] & q^{p_{12}-p_{22}} + q^{p_{22}-p_{12}} \end{pmatrix}. \tag{15}$$

To conclude, the $\mathcal{U}_q(\mathfrak{sl}(2))$ modules with the same value of $h_1 + 2h_2$ that would have the same value of the quadratic Casimir when $q^l = 1$ are pairwise coupled in a single indecomposable representation. The same thing happens in the fusion rule of restricted irreducible representations of $\mathcal{U}_q(\mathfrak{sl}(2))$ [14–16]. The total $\mathcal{U}_q(\mathfrak{sl}(2))$ representation corresponding to a given value of $h_1 + 2h_2$ is actually equivalent, as in the classical or generic case, to the tensor product of two irreducible representations. If the value of $h_1 + 2h_2$ is greater than or equal to $p_{13} - 2p_{23} + p_{33} + 2$ (corresponding to the classical $\mathfrak{sl}(2)$ representation with the highest dimension $p_{13} - p_{33} - 1$), the total $\mathcal{U}_q(\mathfrak{sl}(2))$ representation corresponding to this value is equivalent to

$$j_1 \otimes j_2 \quad \text{with} \quad \begin{cases} j_1 = \frac{1}{2}(p_{13} - p_{23} - 1) \\ j_2 = \frac{1}{6}(p_{13} + p_{23} - 2p_{33} - 1 - (h_1 + 2h_2)) \end{cases} \tag{16}$$

as shown in figure 3 which is easier to understand than the formula. In the case where the value of $h_1 + 2h_2$ is lower or equal to $p_{13} - 2p_{23} + p_{33} + 2$, the total $\mathcal{U}_q(\mathfrak{sl}(2))$ representation is equivalent to

$$j_1 \otimes j_2 \quad \text{with} \quad \begin{cases} j_1 = \frac{1}{2}(p_{23} - p_{33} - 1) \\ j_2 = \frac{1}{6}(h_1 + 2h_2 - (-2p_{13} + p_{23} + p_{33} - 1)) \end{cases} \tag{17}$$

i.e. the same as before, but starting from the opposite edge of the hexagon.

3.2. Other effects of regularization

In the limit when $q^l = 1$, the coefficients that would involve fractions such as $\frac{[0]}{[l]}$ remain zero. Although $[l]$ is zero at $q^l = 1$, there is no ambiguity in such limits. This means in particular that the states that are classically forbidden (those that would not respect the triangular inequalities (7)) remain forbidden at $q^l = 1$. The vector space $V(p_{13}, p_{23}, p_{33})$ on which the representation acts at $q^l = 1$ is the same as in the classical case or at generic q . (As we will see in section 4, the so-obtained representation is sometimes not irreducible, and we will be led to take quotients.)

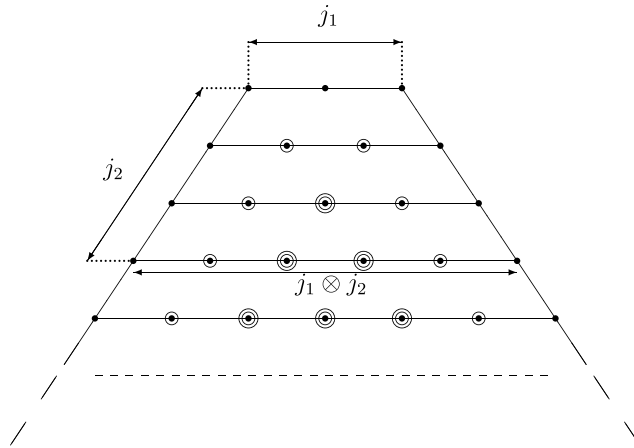


Figure 3. Rule for $\mathcal{U}_q(\mathfrak{sl}(2))$ subrepresentation.

Since redefinition (14) contains coefficients that diverge in the limit $q^l = 1$, we have to carefully check the behaviour of the regularized representation on redefined states and when crossing the boundary of the teepee, the domain that contains the redefined states.

- In appendix A, the actions of f_1 and e_1 are explicitly given. The coefficients are finite. As explained before, they describe well-defined indecomposable representations of $\mathcal{U}_q(\mathfrak{sl}(2))$.

- In appendix B, the actions of e_2 and f_2 within the teepee are computed. The coefficients are also finite. The diverging coefficient in (14) essentially enhances differences of coefficients that have the same limits.

- In appendix C, a compendium of all possible sources of divergences is presented. In each case, the reason why the divergence disappears is briefly explained.

The locations where the generators can lead from a non-redefined state to a redefined one, or vice versa, is the set of G–Z states satisfying one of the following equations:

- left roof: $p_{22} = p_{13} - l$,
- right roof: $p_{12} = p_{33} + l + 1$,
- front ‘entrance’: $p_{22} = p_{11} - l$,
- back ‘entrance’: $p_{12} = p_{11} + l - 1$,
- l -dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$ modules: $p_{12} - p_{22} = l$.

The adjectives ‘left’, ‘right’, ... refer to figure 1. The first four cases are the boundaries of the teepee within the pyramid. The last case corresponds to non-redefined l -dimensional modules. The boundaries are defined as belonging to the teepee. Note that the front and back roofs are boundaries of the pyramid, not boundaries of the teepee in the pyramid.

After the regularization defined by (14), all the coefficients remain finite or go to zero in the limit where $q^l = 1$. A representation $M_q^{\text{reg}}(p_{13}, p_{23}, p_{33})$ at $q^l = 1$ is then obtained by defining the action of the generators using the limit of these coefficients. As these coefficients are finite, they indeed define elements of $\text{End}(V(p_{13}, p_{23}, p_{33}))$. Moreover, these elements satisfy the commutation relations of $\mathcal{U}_q(\mathfrak{sl}(3))$ at $q^l = 1$, since these relations are continuous functions of the coefficients.

4. Reducibility

The regularized representation $M_q^{\text{reg}}(p_{13}, p_{23}, p_{33})$ at $q^l = 1$ is not always irreducible. We recall that we consider $p_{13} - p_{33} > l$ (otherwise, nothing new happens with respect to the generic case).

If p_{23} is equal to $p_{13} - l$ or to $p_{33} + l$, $M_q^{\text{reg}}(p_{13}, p_{23}, p_{33})$ is irreducible. Otherwise, i.e. if $\min(p_{13} - p_{23}, p_{23} - p_{33}) < l$, the application S_2 defined in equation (2) from the vector space $V(p_{33} + l, p_{23}, p_{13} - l)$ to the vector space $V(p_{13}, p_{23}, p_{33})$ is a morphism from the representation $M_q(p_{33} + l, p_{23}, p_{13} - l)$ to the representation $M_q^{\text{reg}}(p_{13}, p_{23}, p_{33})$. Its image is isomorphic to $M_q(p_{33} + l, p_{23}, p_{13} - l)$, and is a subrepresentation of $M_q^{\text{reg}}(p_{13}, p_{23}, p_{33})$. It is easy to check that

- none of the redefined state belongs to this image, and
- no action of the generators connects this image directly to the set of redefined states.

It can then be seen by using only equations (10), (11) that this image really decouples. The factors $[p_{12} - p_{33}(-1)]^{1/2}$ and $[p_{13} - p_{22}(+1)]^{1/2}$, that vanish for $p_{12} = p_{33} + l(+1)$ and $p_{22} = p_{13} - l(-1)$, respectively, are enough.

The representation $M_q^{\text{reg}}(p_{13}, p_{23}, p_{33})$ is then the direct sum of the two subrepresentations respectively characterized by $\max(p_{12} - p_{33}, p_{13} - p_{22} + 1) \geq l$ and $\max(p_{12} - p_{33}, p_{13} - p_{22} + 1) < l$.

The first one, equivalent to $M_q(p_{33} + l, p_{23}, p_{13} - l)$, then has a classical counterpart. In [10], this subrepresentation is identified with the $\min(p_{33} - p_{23} + l, p_{23} - p_{13} + l)$ top layers of the pyramid.

The second one, that contains all the redefined states, has no classical analogue. It corresponds in [10] to the $p_{13} - p_{33} - l$ bottom layers of the pyramid (its height is the same as that of the teepee). We denote it by $M_q^{\text{quot}}(p_{13}, p_{23}, p_{33})$ as it is the quotient of $M_q^{\text{reg}}(p_{13}, p_{23}, p_{33})$ by $S_2(M_q(p_{33} + l, p_{23}, p_{13} - l))$. Its dimension is $d(p_{13}, p_{23}, p_{33}) - d(p_{33} + l, p_{23}, p_{13} - l)$.

These two summands are irreducible themselves. The reducibility of one of the summands would require more singular vectors in the Verma module with the same highest weight as $M_q(p_{13}, p_{23}, p_{33})$ than found in [8].

5. An interesting case: flat representations

An interesting case is provided by the flat representations, i.e. those for which the weight multiplicities are at most 1. They correspond to parameters such that $p_{13} - p_{33} = l + 1$.

In this case, no state needs to be redefined, since the teepee reduces to one single line with l points (with $p_{12} = p_{13}$ and $p_{22} = p_{33} + 1$). These representations are actually integrable in the sense given in section 2, since $\langle \lambda, \theta^\vee \rangle = l - 1$, the maximum value for integrable representations.

If p_{23} is equal to $p_{13} - 1 = p_{33} + l$ or to $p_{13} - l = p_{33} + 1$, then $M_q(p_{13}, p_{23}, p_{33})$ itself is flat and irreducible. Otherwise, the flat irreducible representation is, as explained before, $M_q(p_{13}, p_{23}, p_{33})/S_2(M_q(p_{13} - 1, p_{23}, p_{33} + 1))$, of dimension $d(p_{13}, p_{23}, p_{33}) - d(p_{13} - 1, p_{23}, p_{33} + 1)$.

The states of this quotient are then the G-Z states satisfying the usual triangular inequalities (7) and

$$p_{12} = p_{13} \quad \text{or} \quad p_{22} = p_{33} + 1 = p_{13} - l. \tag{18}$$

The existence and dimension of these representations were known from the character formulae [7, 8].

The flat irreducible representations of $\mathcal{U}_q(\mathfrak{sl}(3))$ were described in [18] as quotients of singular limits of flat periodic representations of dimension l^2 . They were also obtained in [13, 19] within the G–Z basis, but with a different prescription that we will recall now. Consider the vector space spanned by the vectors

$$\left| \begin{array}{ccc} \bar{p}_{13} & \bar{p}_{23} & \bar{p}_{23} - 1 \\ & \bar{p}_{12} & \bar{p}_{23} \\ & & \bar{p}_{11} \end{array} \right\rangle \tag{19}$$

with $\bar{p}_{23} + 2l \geq \bar{p}_{13} > \bar{p}_{23} + l$ and where $\bar{p}_{22} = \bar{p}_{23}$ is frozen.

With the triangular inequalities (7), this defines a $\mathcal{U}_q(\mathfrak{sl}(3))$ representation with a triangular set of weights (hence flat) of dimension $d_0 = d(\bar{p}_{13}, \bar{p}_{23}, \bar{p}_{23} - 1)$. This representation is not irreducible and splits into four subrepresentations obtained as follows:

$$\bar{p}_{13} \geq \bar{p}_{12} > \bar{p}_{23} + l \quad \text{and} \quad \bar{p}_{12} \geq \bar{p}_{11} > \bar{p}_{23} + l.$$

Flat triangular representation (left corner) of dimension $d_1 = d(\bar{p}_{13}, \bar{p}_{23} + l, \bar{p}_{23} + l - 1) = \frac{1}{2}(\bar{p}_{13} - \bar{p}_{23} - l)(\bar{p}_{13} - \bar{p}_{23} - l + 1)$.

$$\bar{p}_{13} \geq \bar{p}_{12} > \bar{p}_{23} + l \quad \text{and} \quad \bar{p}_{12} - l \geq \bar{p}_{11} > \bar{p}_{23}.$$

Another flat triangular representation (right corner), of dimension $d_2 = d(\bar{p}_{13} - l, \bar{p}_{23} - l, \bar{p}_{23} - l - 1) = d_1$.

$$\bar{p}_{13} - l \geq \bar{p}_{12} > \bar{p}_{23} \quad \text{and} \quad \bar{p}_{12} \geq \bar{p}_{11} > \bar{p}_{23}.$$

Another flat triangular representation (bottom corner), of dimension $d_3 = d(\bar{p}_{13} - l, \bar{p}_{23} - l, \bar{p}_{23} - l - 1) = d_1$.

$$\bar{p}_{13} \geq \bar{p}_{12} > \bar{p}_{23} \quad \text{and} \quad \bar{p}_{12} \geq \bar{p}_{11} > \bar{p}_{12} - l \quad \text{and} \quad \bar{p}_{23} + l \geq \bar{p}_{11} > \bar{p}_{23}.$$

The flat hexagonal representation of dimension $d_0 - d_1 - d_2 - d_3 = d_0 - 3d_1$.

This description is linked to the G–Z formalism of this paper by the identification $\bar{p}_{13} = p_{23} + l, \bar{p}_{23} = p_{13} - l, \bar{p}_{33} = p_{33} = p_{13} - l - 1$. A transformation inspired by both S_1 and S_2 relates them, namely

$$\begin{aligned} \left| \begin{array}{ccc} p_{13} & p_{23} & p_{33} \\ & p_{12} & p_{22} \\ & & p_{11} \end{array} \right\rangle &\xrightarrow{\text{if } p_{22} = p_{13} - l} \left| \begin{array}{ccc} p_{23} + l & p_{13} - l & p_{33} \\ & p_{12} & p_{22} \\ & & p_{11} \end{array} \right\rangle \\ \left| \begin{array}{ccc} p_{13} & p_{23} & p_{33} \\ & p_{12} & p_{22} \\ & & p_{11} \end{array} \right\rangle &\xrightarrow{\text{if } p_{12} = p_{13}} \left| \begin{array}{ccc} p_{23} + l & p_{13} - l & p_{33} \\ & p_{22} + l & p_{12} - l \\ & & p_{11} \end{array} \right\rangle \end{aligned} \tag{20}$$

6. Conclusion

The description of $\mathcal{U}_q(\mathfrak{sl}(N))$ representations with the G–Z basis will necessitate the knowledge, within the G–Z basis, of some indecomposable $\mathcal{U}_q(\mathfrak{sl}(N - 1))$ representations, probably those involved in the decomposition of tensor products of irreducible ones [20]. The representations will be built by collecting the $\mathcal{U}_q(\mathfrak{sl}(N - 1))$ representations that would have the same values of the Casimir operators $C_{\mathcal{U}_q(\mathfrak{sl}(N-1))}^{(i)}$ in the limit $q^l = 1$. The use of

transformations that generalize S_1 will help in characterizing them. By doing this, analogues of S_2 will provide the subrepresentations.

The restricted representations we have described here can be used as explained in [19] to build special kinds of partially periodic (unrestricted) irreducible representations of $\mathcal{U}_q(\mathfrak{sl}(N))$ with $N > 3$.

One could also wonder whether the periodic indecomposable representations of $\mathcal{U}_q(\mathfrak{sl}(2))$ of dimension $2l$ that arise in the fusion of periodic irreducible representations [21] may also appear in some $\mathcal{U}_q(\mathfrak{sl}(3))$ irreducible representations.

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Appendix A. Indecomposable $\mathcal{U}_q(\mathfrak{sl}(2))$ subrepresentations

An indecomposable representation of dimension $2l$ is made from the two $\mathcal{U}_q(\mathfrak{sl}(2))$ representations corresponding to $p_{12} \geq p_{11} > p_{22}$ (of dimension $p_{12} - p_{22}$) and $p_{22} + l \geq p_{11} > p_{12} - l$ (of dimension $2l - (p_{12} - p_{22})$). The states that have a common p_{11} are mixed as explained in (14) and the limit $q^l = 1$ is taken. The generators e_1 and f_1 act on it as

$$\begin{aligned}
 f_1 \left| \begin{matrix} p_{12} & & p_{22} \\ & p_{11} & \end{matrix} \right\rangle &= ([p_{12} - p_{11} + 1][p_{11} - p_{22} - 1])^{1/2} \left| \begin{matrix} p_{12} & & p_{22} \\ & p_{11} - 1 & \end{matrix} \right\rangle \\
 &\text{for } p_{11} > p_{22} + l + 1 \quad \text{or} \quad p_{11} \leq p_{12} - l \\
 f_1 \left| \begin{matrix} p_{12} & & p_{22} \\ & p_{11} & \end{matrix} \right\rangle' &= ([p_{12} - p_{11} + 1][p_{11} - p_{22} - 1])^{1/2} \left| \begin{matrix} p_{12} & & p_{22} \\ & p_{11} - 1 & \end{matrix} \right\rangle' \\
 &\text{for } p_{22} + l \geq p_{11} > p_{12} - l + 1 \\
 f_1 \left| \begin{matrix} p_{12} & & p_{22} \\ & p_{22} + l + 1 & \end{matrix} \right\rangle &= ([p_{12} - p_{22} - l])^{1/2} \left| \begin{matrix} p_{12} & & p_{22} \\ & p_{22} + l & \end{matrix} \right\rangle' \\
 &\text{(instead of 0)} \\
 f_1 \left| \begin{matrix} p_{12} & & p_{22} \\ & p_{12} - l + 1 & \end{matrix} \right\rangle' &= 0 \\
 f_1 \left| \begin{matrix} p_{22} + l & & p_{12} - l \\ & p_{12} - l + 1 & \end{matrix} \right\rangle' &= (-[p_{12} - p_{22} - l])^{1/2} \left| \begin{matrix} p_{12} & & p_{22} \\ & p_{12} - l & \end{matrix} \right\rangle' \\
 &\text{(instead of 0)} \\
 f_1 \left| \begin{matrix} p_{22} + l & & p_{12} - l \\ & p_{11} & \end{matrix} \right\rangle' &= ([p_{12} - p_{11} + 1][p_{11} - p_{22} - 1])^{1/2} \\
 &\quad \times \left| \begin{matrix} p_{22} + l & & p_{12} - l \\ & p_{11} - 1 & \end{matrix} \right\rangle' \\
 &\quad + (-[p_{12} - p_{11} + 1][p_{11} - p_{22} - 1])^{-1/2} [p_{12} - p_{22}] \\
 &\quad \times \left| \begin{matrix} p_{12} & & p_{22} \\ & p_{11} - 1 & \end{matrix} \right\rangle' \\
 &\text{for } p_{22} + l \geq p_{11} > p_{12} - l + 1.
 \end{aligned} \tag{21}$$

$$\begin{aligned}
 e_1 \left| \begin{array}{cc} p_{12} & p_{22} \\ p_{11} & \end{array} \right\rangle &= ([p_{12} - p_{11}][p_{11} - p_{22}])^{1/2} \left| \begin{array}{cc} p_{12} & p_{22} \\ p_{11} + 1 & \end{array} \right\rangle \\
 &\text{for } p_{11} > p_{22} + l \text{ or } p_{11} < p_{12} - l \\
 e_1 \left| \begin{array}{cc} p_{12} & p_{22}' \\ p_{11} & \end{array} \right\rangle &= ([p_{12} - p_{11}][p_{11} - p_{22}])^{1/2} \left| \begin{array}{cc} p_{12} & p_{22}' \\ p_{11} + 1 & \end{array} \right\rangle \\
 &\text{for } p_{22} + l > p_{11} > p_{12} - l \\
 e_1 \left| \begin{array}{cc} p_{12} & p_{22}' \\ p_{12} - l & \end{array} \right\rangle &= ([p_{12} - p_{22} - l])^{1/2} \left| \begin{array}{cc} p_{12} & p_{22}' \\ p_{12} - l + 1 & \end{array} \right\rangle \\
 &\text{(instead of 0)} \\
 e_1 \left| \begin{array}{cc} p_{12} & p_{22}' \\ p_{22} + l & \end{array} \right\rangle &= 0 \tag{22} \\
 e_1 \left| \begin{array}{cc} p_{22} + l & p_{12} - l \\ p_{22} + l & \end{array} \right\rangle &= (-[p_{12} - p_{22} - l])^{1/2} \left| \begin{array}{cc} p_{12} & p_{22}' \\ p_{22} + l + 1 & \end{array} \right\rangle \\
 &\text{(instead of 0)} \\
 e_1 \left| \begin{array}{cc} p_{22} + l & p_{12} - l \\ p_{11} & \end{array} \right\rangle &= ([p_{12} - p_{11}][p_{11} - p_{22}])^{1/2} \left| \begin{array}{cc} p_{22} + l & p_{12} - l \\ p_{11} + 1 & \end{array} \right\rangle \\
 &\quad + (-[p_{12} - p_{11}][p_{11} - p_{22}])^{-1/2} [p_{12} - p_{22}] \left| \begin{array}{cc} p_{12} & p_{22}' \\ p_{11} + 1 & \end{array} \right\rangle \\
 &\text{for } p_{22} + l > p_{11} > p_{12} - l.
 \end{aligned}$$

We can check that e_1^l and f_1^l vanish on this indecomposable representation.

Appendix B. Action of e_2 and f_2 in the teepee

Let us consider $\left| \begin{array}{cc} p_{12} & p_{22}' \\ p_{11} & \end{array} \right\rangle$ with $p_{12} - p_{22} > l$, such that both (7) and (13) are satisfied. Then $f_2 \left| \begin{array}{cc} p_{12} & p_{22}' \\ p_{11} & \end{array} \right\rangle$ is given by a formula analogous to (11), but with primed states on the right-hand side. This is true as long as the final primed states are defined. One has

$$\begin{aligned}
 f_2 \left| \begin{array}{cc} p_{22} + l & p_{12} - l \\ p_{11} & \end{array} \right\rangle &= \left(\frac{P_1 P_2}{P_3}(1, 2; p) \right)^{1/2} \left| \begin{array}{cc} p_{22} + l & p_{12} - l - 1 \\ p_{11} & \end{array} \right\rangle \\
 &\quad + \left(\frac{P_1 P_2}{P_3}(2, 2; p_{22}) \right)^{1/2} \left| \begin{array}{cc} p_{22} + l - 1 & p_{12} - l \\ p_{11} & \end{array} \right\rangle \\
 &\quad + \mathcal{D}_{\substack{p_{12} \rightarrow p_{12} - l \\ p_{22} \rightarrow p_{22} + l}} \left(-\frac{P_1 P_2}{P_3}(1, 2; p) \right)^{1/2} \left| \begin{array}{cc} p_{12} - 1 & p_{22}' \\ p_{11} & \end{array} \right\rangle \\
 &\quad + \mathcal{D}_{\substack{p_{12} \rightarrow p_{12} - l \\ p_{22} \rightarrow p_{22} + l}} \left(-\frac{P_1 P_2}{P_3}(2, 2; p) \right)^{1/2} \left| \begin{array}{cc} p_{12} & p_{22} - 1 \\ p_{11} & \end{array} \right\rangle \tag{23} \\
 e_2 \left| \begin{array}{cc} p_{22} + l & p_{12} - l \\ p_{11} & \end{array} \right\rangle &= \left(\frac{P_1 P_2}{P_3}(1, 2; p_{12} + 1) \right)^{1/2} \left| \begin{array}{cc} p_{22} + l & p_{12} - l + 1 \\ p_{11} & \end{array} \right\rangle \\
 &\quad + \left(\frac{P_1 P_2}{P_3}(2, 2; p_{22} + 1) \right)^{1/2} \left| \begin{array}{cc} p_{22} + l + 1 & p_{12} - l \\ p_{11} & \end{array} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 & + \mathcal{D}_{\substack{p_{12} \rightarrow p_{12}-l \\ p_{22} \rightarrow p_{22}+l}} \left(-\frac{P_1 P_2}{P_3} (1, 2; p_{12} + 1) \right)^{1/2} \left| \begin{array}{cc} p_{12} + 1 & p_{22} \\ & p_{11} \end{array} \right\rangle \\
 & + \mathcal{D}_{\substack{p_{12} \rightarrow p_{12}-l \\ p_{22} \rightarrow p_{22}+l}} \left(-\frac{P_1 P_2}{P_3} (2, 2; p_{22} + 1) \right)^{1/2} \left| \begin{array}{cc} p_{12} & p_{22} + 1 \\ & p_{11} \end{array} \right\rangle
 \end{aligned} \tag{24}$$

where

$$\mathcal{D}_{a \rightarrow b}(f) = \lim_{q^l \rightarrow 1} \frac{1}{[l]} (f(a) - f(b)). \tag{25}$$

In the cases we consider, arguments a and b differ by multiples of l and the limit in (25) is finite. Moreover, \mathcal{D} acts as a derivative and $\mathcal{D}_{\substack{a \rightarrow b \\ c \rightarrow d}}(f) = \mathcal{D}_{a \rightarrow b}(f) + \mathcal{D}_{c \rightarrow d}(f)$.

Appendix C. List of all the possible divergences

C.1. Vanishing denominators in equation (11)

It is easy to see that the denominators in the action of e_2 and f_2 (11) vanish in the following cases.

- For a ‘classical’ reason, i.e. when $p_{12} - p_{22} = 1$ and when the action of e_2 or f_2 would lead to a forbidden state where $p_{12} - p_{22} = 0$. In such cases, the denominator comes with two zeros in the numerator that cancel this branching.
- When acting on a G–Z state with $p_{12} - p_{22} = l$. The two resulting states with diverging coefficients actually belong to the set of redefined states. The regularization compensates the divergence. In the case when one of the resulting states does not exist classically, the coefficient of the single remaining state (hence not to be redefined) has also a zero in the numerator and it remains finite.
- When the action leads to a state with $p_{12} - p_{22} = l$. If only one initial state can lead to it, the coefficient is finite due to a vanishing numerator. If two states lead to it, they have the same weight *and* same value of $C_{U_q(sl(2))}$, so they are redefined. The action on these redefined states contains finite differences of the diverging coefficients.

C.2. Entering and leaving the teepee

We now summarize the reasons why the actions of f_2 and e_2 are well-defined on the boundary of the teepee. Let us first consider the action of f_2 and e_2 on a state lying outside the teepee, the effect of which is to enter the teepee. We have four different ways of entering.

- Through the left roof: $p_{22} = p_{13} - l$ for the final state. This is reached as f_2 acts on $\left| \begin{array}{cc} p_{12} & p_{13} - l + 1 \\ & p_{11} \end{array} \right\rangle$. A vanishing factor $[p_{13} - p_{22}]$ from the numerator P_1 compensates the diverging factor from the redefinition (14) of the final state. This is true unless $p_{12} = p_{13}$, in which case this vanishing factor compensates a factor from P_3 that goes to zero. We arrive in this case on $p_{12} - p_{22} = l$ and there is no redefinition.
- Through the right roof: $p_{12} = p_{33} + l + 1$ for the final state. This is reached as e_2 acts on $\left| \begin{array}{cc} p_{33} + l & p_{22} \\ & p_{11} \end{array} \right\rangle$. A vanishing factor $[p_{12} - p_{33} - 1]$ from the numerator P_1 compensates the diverging factor from the redefinition (14) of the final state. This is true unless $p_{22} = p_{33} + 1$, in which case this vanishing factor compensates a factor from P_3 that goes to zero. We arrive in this case on $p_{12} - p_{22} = l$ and there is no redefinition.

• Through the front ‘entrance’: $p_{22} = p_{11} - l$ for the final state. This is reached as e_2 acts on $\left| \begin{array}{cc} p_{12} & p_{11} - l - 1 \\ & p_{11} \end{array} \right\rangle$, with $p_{12} - p_{11} > 0$. A vanishing factor $[p_{11} - p_{22}]$ from the numerator P_2 compensates the diverging factor from the redefinition (14) of the final state. Note that for $p_{12} = p_{11}$, this final state is not redefined, and the compensation comes from the denominator.

• Through the back ‘entrance’: $p_{12} = p_{11} + l - 1$ for the final state. This is reached as f_2 acts on $\left| \begin{array}{cc} p_{11} + l & p_{12} \\ & p_{11} \end{array} \right\rangle$, with $p_{11} - p_{22} > 1$. A vanishing factor $[p_{12} - p_{11} + 1]$ from the numerator P_2 compensates the diverging factor from the redefinition (14) of the final state. Note that for $p_{11} = p_{22} + 1$, this final state is not redefined, and the compensation comes from the denominator.

We now consider f_2 and e_2 acting on a redefined state, such that these actions lead to at least one non-redefined state. Again, the diverging coefficient involved in (14) may be a source of a problem in the boundary. Without going into details, the finiteness argument is again that the boundary of the teepee is a place where one of the numerators, P_1 or P_2 , vanishes and compensates the denominator in (14).

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